JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 51, 695–704 (1975)

Dynamic Programming and Duality in Linear Programming

D. J. WHITE

Department of Decision Theory, University of Manchester, Manchester M13 9PL, England

Submitted by Richard Bellman

INTRODUCTION

The symmetric primal form of linear programming is as follows.

$$P_n(\mathbf{z}): \quad \text{maximize} \left\{ \sum_{j=1}^{n} v_j x_j \right\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant z_i, \qquad i = 1, 2, \ldots, m, \tag{2}$$

$$x_j \geqslant 0, \qquad j = 1, 2, \ldots, n. \tag{3}$$

The dual form is as follows.

$$D_n(\mathbf{z}): \quad \text{minimize} \left\{ \sum_{i=1}^{m} z_i \pi_i \right\} \tag{4}$$

subject to

$$\sum_{i=1}^{m} a_{ij} \pi_i \geqslant v_j, \qquad j = 1, 2, \ldots, n, \tag{5}$$

$$\pi_i \geqslant 0, \qquad i = 1, 2, \ldots, m. \tag{6}$$

There are certain well-known relationships between $P_n(\mathbf{z})$ and $D_n(\mathbf{z})$. The purpose of this note is to show how these relationships can be established using dynamic programming.

Dreyfus and Freimer [1] use a dynamic programming approach to the duality theorem. However this is heuristic because it makes differentiability assumptions which are not examined. In addition, the cases of infeasibility and unboundedness are not examined.

In tackling this problem use is made of "supremum" and "infimum," which reduce to the usual "maximum" and "minimum" when finite optima exist. This facilitates the analysis and makes the derivation of the full theorem fairly simple once the central optimization equivalence of the primal and dual problems has been established in one form.

It is to be noted that the proof requires no special attention to degeneracy conditions which can complicate normal proofs.

Having established the duality theorem it is possible to provide equivalent mixed forms of the primal and dual problems. These provide the possibility of obtaining a variety of equivalent linear programming forms.

Dynamic Programming Formulation

Using Bellman [1] we have the following formulation.

$$P_k(\mathbf{z}):$$

$$k \geqslant 1 \tag{7}$$

$$f_k(\mathbf{z}) = \sup_{x \geqslant 0} \{v_k x + f_{k-1}(\mathbf{z} - x\mathbf{a}_k)\},$$

where

$$\mathbf{a}_k \text{ is the column } (a_{1k}, a_{2k}, \ldots, a_{mk}), \tag{8}$$

$$f_k(\mathbf{z}) \text{ is the supremum value of } \sum_{j=1}^{k} v_j x_j$$

subject to

$$\sum_{j=1}^{k} a_{ij} x_j \leqslant z_i, \qquad i = 1, 2, \ldots, m, \tag{9}$$

$$x_j \geqslant 0, \qquad j = 1, 2, \ldots, k. \tag{10}$$

It is important to notice that in expression (7) we only constrain x to be nonnegative. The reason for this is that if any x gives rise to infeasibilities at later stages, this will be reflected in the value of the function $f_{k-1}(\cdot)$.

We similarly define $F_k(\mathbf{z})$ for $D_k(\mathbf{z})$.

We need to consider $k = 0$ separately.

For general functions $g(\mathbf{t})$ defined over a region R, if R is infeasible we put

$$\inf_{\mathbf{t} \in R} \{g(\mathbf{t})\} = \infty, \qquad \sup_{\mathbf{t} \in R} \{g(\mathbf{t})\} = -\infty \tag{11}$$

and wherever one of the results (11) arise we equate this with R being infeasible.

For $k = 0$, we have the primal problem as follows.

$$f_0(\mathbf{z}) = 0 \qquad \text{if} \quad z_i \geqslant 0, \qquad i = 1, 2,..., m \tag{12}$$

If $P_0(z)$ is infeasible, then

$$f_0(\mathbf{z}) = -\infty \tag{13}$$

For the dual problem we have the following.

$$F_0(\mathbf{z}) = \inf_{\{\pi_i\} \geqslant 0} \left\{ \sum_{i=1}^{m} \pi_i z_i \right\}. \tag{14}$$

We easily see that $f_0(\mathbf{z}) = F_0(\mathbf{z})$.

DUALITY THEOREM.

$$(\alpha) \quad f_k(\mathbf{z}) = F_k(\mathbf{z}),\ k \geqslant 0 \tag{15}$$

$$(\beta) \quad P_k(\mathbf{z})\ \textit{feasible},\ D_k(\mathbf{z})\ \textit{infeasible} \rightleftarrows f_k(\mathbf{z}) = \infty,\ k \geqslant 1 \tag{16}$$

$$D_k(\mathbf{z})\ \textit{feasible},\ P_k(\mathbf{z})\ \textit{infeasible} \rightleftarrows F_k(\mathbf{z}) = -\infty,\ k \geqslant 1 \tag{17}$$

$$(\gamma) \quad P_k(\mathbf{z})\ \textit{feasible},\ D_k(\mathbf{z})\ \textit{feasible} \rightleftarrows x_t{}^*(v_k - \boldsymbol{\pi}_k{}^*\mathbf{a}_k) = 0,\ 1 \leqslant t \leqslant k, \tag{18}$$

where $x_t{}^*$, $\boldsymbol{\pi}_k{}^*$ are the optimum values of the variables $\{x_t\}$ for $P_k(\mathbf{z})$ and optimum solutions for $D_k(\mathbf{z})$, respectively.

A similar result applies when the dual and primal roles are reversed.

The central part of the theorem is part (α) and is true irrespective of feasibility and boundedness considerations. The rest follows very easily from this.

It will be seen that by putting $k = n$, we have the normal statement of the theorem, if we put $t = k$ in (18).

We first of all establish a lemma.

LEMMA. *Let $\phi(\mathbf{x}, \boldsymbol{\pi})$ be concave in finite dimensional vectors $\mathbf{x}$ and convex in finite dimensional vectors π over nonempty convex spaces X, Π, respectively, and let $\phi(x, \pi)$ be continuous over $X \times \pi$. Then*

$$\inf_{\boldsymbol{\pi} \in \Pi} \sup_{\mathbf{x} \in X} \{\phi(\mathbf{x}, \boldsymbol{\pi})\} = \sup_{\mathbf{x} \in X} \inf_{\boldsymbol{\pi} \in \Pi} \{\phi(\mathbf{x}, \boldsymbol{\pi})\}. \tag{19}$$

Proof. In general we have

$$\inf_{\pi\in\Pi}\sup_{x\in X}\{\phi(\mathbf{x},\boldsymbol{\pi})\} \geqslant \sup_{x\in X}\inf_{\pi\in\Pi}\{\phi(\mathbf{x},\boldsymbol{\pi})\}. \tag{20}$$

Let S, T be feasible bounded closed convex subsets of X, Π, respectively. Then (see Stoer and Witzgall [2])

$$\inf_{\pi\in T}\sup_{x\in S}\{\phi(\mathbf{x},\boldsymbol{\pi})\} = \sup_{x\in S}\inf_{\pi\in T}\{\phi(\mathbf{x},\boldsymbol{\pi})\}. \tag{21}$$

Also,

$$\inf_{\pi\in T}\sup_{x\in S}\{\phi(\mathbf{x},\boldsymbol{\pi})\} \geqslant \inf_{\pi\in\Pi}\sup_{x\in S}\{\phi(\mathbf{x},\boldsymbol{\pi})\} \tag{22}$$

$$\sup_{x\in S}\inf_{\pi\in T}\{\phi(\mathbf{x},\boldsymbol{\pi})\} \leqslant \sup_{x\in X}\inf_{\pi\in T}\{\phi(\mathbf{x},\boldsymbol{\pi})\}. \tag{23}$$

Hence, combining (21), (22), and (23) we have

$$\inf_{\pi\in\Pi}\sup_{x\in S}\{\phi(\mathbf{x},\boldsymbol{\pi})\} \leqslant \sup_{x\in X}\inf_{\pi\in T}\{\phi(\mathbf{x},\boldsymbol{\pi})\}. \tag{24}$$

Then from (24) we have

$$\begin{aligned}\inf_{\pi\in\Pi}\sup_{x\in X}\{\phi(\mathbf{x},\boldsymbol{\pi})\} &= \inf_{\pi\in\Pi}\sup_{S\subseteq X}\sup_{x\in S}\{\phi(\mathbf{x},\boldsymbol{\pi})\}\\ &\leqslant \sup_{x\in X}\inf_{T\subseteq\Pi}\inf_{\pi\in T}\{\phi(\mathbf{x},\boldsymbol{\pi})\}\\ &= \sup_{x\in X}\inf_{\pi\in\Pi}\{\phi(\mathbf{x},\boldsymbol{\pi})\}.\end{aligned} \tag{25}$$

Inequalities (20) and (25) establish the lemma.

Note that this lemma requires no boundedness constraints on $\phi(\mathbf{x},\boldsymbol{\pi})$ nor on X, Π.

Proof of Theorem. Part (α) of the theorem has been established as true for $k = 0$. Let us now assume that part (α) of the theorem is true for $k - 1$, $k - 2,\ldots, 0$. We have four cases to consider.

Case A. $P_k(\mathbf{z})$ and $D_k(\mathbf{z})$ feasible.

Case B. $P_k(\mathbf{z})$ feasible, $D_k(\mathbf{z})$ infeasible.

Case C. $P_k(\mathbf{z})$ infeasible, $P_k(\mathbf{z})$ feasible.

Case D. $P_k(\mathbf{z})$, $D_k(\mathbf{z})$ infeasible.

In what follows we will drop suffixes from x_k and $\boldsymbol{\pi}_k$ for convenience.

Case A. Let Π_k be the feasible set for $D_k(z)$.

$$f_k(\mathbf{z}) = \sup_{x \geqslant 0} \{v_k x + f_{k-1}(\mathbf{z} - x\mathbf{a}_k)\} \tag{26}$$

$$= \sup_{x \geqslant 0} \{v_k x + \inf_{\boldsymbol{\pi} \in \Pi_{k-1}} \{\boldsymbol{\pi}(\mathbf{z} - x\mathbf{a}_k)\}\} \tag{27}$$

$$= \sup_{x \geqslant 0} \inf_{\boldsymbol{\pi} \in \Pi_{k-1}} \{\boldsymbol{\pi}\mathbf{z} + x(v_k - \boldsymbol{\pi}\mathbf{a}_k)\} \tag{28}$$

$$= \inf_{\boldsymbol{\pi} \in \Pi_{k-1}} \sup_{x \geqslant 0} \{\boldsymbol{\pi}\mathbf{z} + x(v_k - \boldsymbol{\pi}\mathbf{a}_k)\} \qquad \text{(using the lemma)} \tag{29}$$

$$= \inf_{\boldsymbol{\pi} \in \Pi_{k-1}} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}. \tag{30}$$

Then

$$f_k(\mathbf{z}) = \min\{\inf_{\boldsymbol{\pi} \in \Pi_k} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}; \inf_{\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}\}. \tag{31}$$

Now, if $\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k$,

$$\sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\} \text{ is given by } x = \infty,$$

and hence

$$\inf_{\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\} = \infty. \tag{32}$$

We can, therefore ignore $\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k$ and

$$f_k(\mathbf{z}) = \inf_{\boldsymbol{\pi} \in \Pi_k} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}. \tag{33}$$

If $E_k \equiv \{\boldsymbol{\pi} \in \Pi_k$ such that $v_k = \boldsymbol{\pi}\mathbf{a}_k$, then

$$f_k(\mathbf{z}) = \min\{\inf_{\boldsymbol{\pi} \in E_k} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}; \inf_{\boldsymbol{\pi} \in \Pi_k - E_k} \sup_{x \geqslant 0} \{\theta_k(\mathbf{z}, x, \boldsymbol{\pi})\}\} \tag{34}$$

$$= \min\{\inf_{\boldsymbol{\pi} \in E_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\}; \inf_{\boldsymbol{\pi} \in \Pi_k - E_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\}\}$$

$$= \inf_{\boldsymbol{\pi} \in \Pi_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\} = F_k(\mathbf{z}) \tag{35}$$

and part (α) of the theorem is true for k.

Case B. Expression (28) is valid and Π_{k-1} is feasible. Since Π_k is infeasible, the only feasible $\boldsymbol{\pi}$ satisfy $v_k > \boldsymbol{\pi} \cdot \mathbf{a}_k$. Using the lemma we see that

$$\begin{aligned} f_k(\mathbf{z}) &= \sup_{x \geqslant 0} \inf_{\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k} \{\boldsymbol{\pi}\mathbf{z} + x(v_k - \boldsymbol{\pi}\mathbf{a}_k)\} \\ &= \inf_{\boldsymbol{\pi} \in \Pi_{k-1} - \Pi_k} \sup_{x \geqslant 0} \{\boldsymbol{\pi}\mathbf{z} + x(v_k - \boldsymbol{\pi}\mathbf{a}_k)\} \\ &= \infty \\ &= \inf_{\boldsymbol{\pi} \in \Pi_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\} = F_k(\mathbf{z}) \text{ by convention} \end{aligned} \tag{36}$$

and part (α) of the theorem is true for k.

Case C. We simply interchange the roles of the problems $P_k(\mathbf{z})$, $D_k(\mathbf{z})$ to produce

$$F_k(\mathbf{z}) = \inf_{\boldsymbol{\pi} \in \Pi_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\} = -\infty = f_k(\mathbf{z}) \text{ by convention} \tag{37}$$

and part (α) of the theorem is true for k.

Case D. This is excluded.

We have thus shown in each of the three meaningful Cases A, B, C that

$$f_k(\mathbf{z}) = \inf_{\boldsymbol{\pi} \in \Pi_k} \{\boldsymbol{\pi} \cdot \mathbf{z}\} \tag{38}$$

and part (α) of the theorem is true for k.

Part (β) comes directly from the previous analysis of Cases B and C.

Hence parts (α) and (β) of the theorem are true for all $k \geqslant 1$, since part (α) of the theorem is true for $k = 0$.

Let us now consider part (γ) of the theorem, which relates only to Case A.

In Case A there will exist finite optimum solutions $\boldsymbol{\pi}^*$ and x^* and from expression (29) we obtain

$$f_k(\mathbf{z}) = \sup_{x \geqslant 0} \{\boldsymbol{\pi}^*\mathbf{z} + x(v_k - \boldsymbol{\pi}^*\mathbf{a}_k)\}. \tag{39}$$

Hence, unless $v_k = \boldsymbol{\pi}^*\mathbf{a}_k$, we would have $x = 0$.

Now, with $k \geqslant 2$, referring to expressions (26), (27), (28), (29), and (38), we see that if $\boldsymbol{\pi}^*$ is optimum for $D_k(\mathbf{z})$ it is also optimum for $D_{k-1}(\mathbf{z} - x^*\mathbf{a}_k)$. Repeating the argument we see that $\boldsymbol{\pi}^*$ will be optimum for all residual duals when modifications to the constraint vector $\mathbf{z}$ have been made in the light of decisions up to that stage. Hence unless, for $1 \leqslant t \leqslant k$, $v_t = \boldsymbol{\pi}^*\mathbf{a}_t$, we have $x_t^* = 0$. (This can also easily be seen from expression (43) of the next section in a more direct manner.)

Thus part (γ) of the theorem is established for $k \geqslant 1$.

The main theorem is now complete. Note that when both optimum solutions are finite we may replace "sup" and "inf" by "max" and "min," respectively.

The Mixed Primal Dual Form

Let us now restrict ourselves to Case A.

Let $X_{k,s+1}(\mathbf{z})$ be the set of $(x_k, x_{k-1}, \ldots, x_{s+1})$, for which $P_k(\mathbf{z})$ has a feasible solution.

We may then extend expression (28) to

$$f_k(\mathbf{z}) = \sup_{\substack{x_t \geqslant 0 \\ k \geqslant t \geqslant s+1}} \inf_{\boldsymbol{\pi} \in \Pi_s} \left\{ \boldsymbol{\pi} \cdot \mathbf{z} + \sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \right\}. \tag{40}$$

By a similar argument to that of the earlier text, we also have

$$f_k(\mathbf{z}) = \inf_{\boldsymbol{\pi} \in \Pi_s} \sup_{\substack{x_t \geqslant 0 \\ k \geqslant t \geqslant s+1}} \left\{ \boldsymbol{\pi} \cdot \mathbf{z} + \sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \right\} \tag{41}$$

$$= \inf_{\boldsymbol{\pi} \in \Pi_s} \{ V_{k,s}(\boldsymbol{\pi}) \}. \tag{42}$$

By a further similar argument to that of the earlier text, we see that

$$f_k(\mathbf{z}) = \inf_{\boldsymbol{\pi} \in \Pi_k} \sup_{\substack{x_t \geqslant 0 \\ k \geqslant t \geqslant s+1}} \left\{ \boldsymbol{\pi} \cdot \mathbf{z} + \sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \right\}. \tag{43}$$

Now from (42) we have, for $\boldsymbol{\pi} \in \Pi_s$:

$$\boldsymbol{\pi} \cdot \mathbf{z} + \sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \leqslant V_{k,s}(\boldsymbol{\pi}), \qquad \forall \{x_t\} \geqslant 0. \tag{44}$$

We thus have a mixed primal dual form, viz

$$\inf_{\boldsymbol{\pi} \in \Pi_s} \{V\} \tag{45}$$

subject to

$$\boldsymbol{\pi} \cdot \mathbf{z} + \sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \leqslant V, \qquad \forall \{x_t\} \geqslant 0, \quad k \geqslant t \geqslant s+1. \tag{46}$$

Let us suppose that M is an upper bound for the feasible values of $\sum_{t=s+1}^{k} x_t$ given $\mathbf{z}$. Then we require

$$\boldsymbol{\pi} \cdot \mathbf{z} + M(v_t - \boldsymbol{\pi} \cdot \mathbf{a}_t) \leqslant V_{ks}(\boldsymbol{\pi}), \qquad \forall k \geqslant t \geqslant s + 1 \tag{47}$$

$$\boldsymbol{\pi} \cdot \mathbf{z} \leqslant V_{ks}(\boldsymbol{\pi}) \tag{48}$$

Since any vector $[x_t]$, $k \geqslant t \geqslant s + 1$ can be put in the form

$$[x_t] = \lambda_0 \mathbf{0} + \sum_{t=s+1}^{k} \lambda_t \boldsymbol{\delta}_t M, \qquad \lambda_0 + \sum_{t=s+1}^{k} \lambda_t = 1, \qquad \lambda_j \geqslant 0, \quad \forall j, \tag{49}$$

where $\mathbf{0}$ is a $(k - s)$ dimensional zero vector and $\boldsymbol{\delta}_t$ is a $(k - s)$ dimensional vector with unity in the position corresponding to t and zeros elsewhere, we see that a necessary and sufficient condition for (44) to hold is that (47) and (48) hold.

The equivalent problem now becomes

$$\inf_{\boldsymbol{\pi} \in \Pi_s} \{V\} \tag{50}$$

subject to

$$\boldsymbol{\pi}(\mathbf{z} - M\mathbf{a}_t) + Mv_t \leqslant V, \qquad k \geqslant t \geqslant s + 1 \tag{51}$$

$$\boldsymbol{\pi} \cdot \mathbf{z} \leqslant V. \tag{52}$$

This problem has $k + 1$ inequalities, in addition to $\pi_i \geqslant 0$, $i = 1, 2,..., m$, whereas $D_k(\mathbf{z})$ has only k inequalities in addition to $\pi_i \geqslant 0$, $i = 1, 2,..., m$.

A similar treatment for expression (40) gives an equivalent problem:

$$\sup_{\substack{x_t \geqslant 0 \\ k \geqslant t \geqslant s+1}} \{V\} \tag{53}$$

subject to

$$\sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi}\mathbf{a}_t) + \boldsymbol{\pi} \cdot \mathbf{z} \geqslant V \qquad \forall \boldsymbol{\pi} \in \Pi_s . \tag{54}$$

The analogous formulation to (50), (51), and (52) is

$$\sup_{\substack{x_t \geqslant 0 \\ k \geqslant t \geqslant s+1}} \{V\} \tag{55}$$

subject to

$$\sum_{t=s+1}^{k} x_t(v_t - \boldsymbol{\pi}^r\mathbf{a}_t) + \boldsymbol{\pi} \cdot \mathbf{z} \geqslant V, \qquad \forall \boldsymbol{\pi}^r, \tag{56}$$

where $\{\boldsymbol{\pi}^r\}$ are a finite set of generators of the convex set Π_s .

In the special case of $s = 0$, (48), (49), and (50) become

$$\inf\{V\} \tag{57}$$

subject to

$$\pi(\mathbf{z} - M\mathbf{a}_t) + Mv_t \leqslant V, \qquad k \geqslant t \geqslant 1 \tag{58}$$

$$\pi \cdot \mathbf{z} \leqslant V \tag{59}$$

$$\pi_i \geqslant 0, \qquad i = 1, 2, \ldots, m, \tag{60}$$

and (53), (54) become

$$\sup\{V\} \tag{61}$$

subject to

$$\sum_{t=1}^{k} x_t(v_t - La_{it}) + Lz_i \geqslant V, \qquad i = 1, 2, \ldots, m. \tag{62}$$

$$\sum_{t=1}^{k} x_t v_t \geqslant V \tag{63}$$

$$x_t \geqslant 0, \qquad k \geqslant t \geqslant 1, \tag{64}$$

where L is an upper bound for the feasible values of $\sum_{i=1}^{m} \pi_i$.

Expressions (45), (46) give a form of mixed primal dual. Expressions (53), (54), and (55), (56) give a further form of mixed primal dual. These are not symmetrical expressions. If we reverse the role of the primal and the duals we obtain symmetrical analogs of (45), (46) and of (53), (54).

The expressions (45), (46), and (53), (54), and (55), (56) and their symmetrical analogs, do not raise much hope of computational aids.

Expressions (50), (51), and (52) and their symmetrical analogs provide new formulations where advantages may arise because of the new coefficients involved, particularly since there is some freedom in the choice of M and L.

Although (57), (58), (59), (60), and (61), (62), (63), (64) are symmetric forms, they are not the duals of each other, thus giving rise to further possibilities by taking the duals of these.

When $M, L \to \infty$, we obtain the original primal and dual problems.

It will, of course, be true that for the optimal solutions, inequalities (52), (59), and (63) will become equalities, and that (50), (51), and (52) reduce to $D_k(\mathbf{z})$, (57), (58), (59), and (60) reduce to $D_k(\mathbf{z})$ and (61), (62), (63), and (64) reduce to $P_k(\mathbf{z})$. Nevertheless these representations of the problems are different and will have different computational aspects.

References

1. R. Bellman and S. Dreyfus, "Applied Dynamic Programming," Princeton University Press, 1962.
2. J. Stoer and C. Witzgall, "Convexity and Optimisation in Finite Dimensions, Vol. I," Springer-Verlag, New York/Berlin, 1970.